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THE RANGE OF A PROJECTION OF SMALL NORM IN l_1^n

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ABSTRACT

It is proved that there exists a positive function $\phi(\varepsilon)$ defined for sufficiently small $\varepsilon > 0$ and satisfying $\lim_{\varepsilon\to 0}\phi(\varepsilon) = 0$ such that for any integers $n > k > 0$, if Q is a projection of l_1^n onto a k-dimensional subspace E with $\|Q\| \leqq 1 + \varepsilon$ then there is an integer $h \geqq k(1 - \phi(\varepsilon))$ and an h-dimensional subspace F of E with $d(F, l_1^h) \leqq 1 + \phi(\varepsilon)$ where $d(X, Y)$ denotes the Banach–Mazur distance between the Banach spaces X and Y. Moreover, there is a projection P of l_1^n onto F with $\|P\| \leqq 1 + \phi(\varepsilon)$.

1. Introduction

A Banach space X is called a P_λ space if whenever X is contained in a Banach space Y there is a projection P of Y onto X with $\|P\| < \lambda$. It is well known and easy to see that for each set M, the space $l_\infty(M)$ (= the space of all bounded real functions f on M with $\|f\| = \sup_{m\in M}|f(m)|$) is a P_1 space. Nachbin [4], Goodner [2] and Kelley [3] characterized the P_1 spaces. They showed that X is a P_1 space if and only if X is isometric to a space $C(S)$ where S is compact, Hausdorff and extremally disconnected. In particular, every finite dimensional P_1 space is isometric to l_∞^n (= the space of n-tuples $x = (x_1, x_2, \cdots, x_n)$ of real numbers with $\|x\| = \max_{1\leqq i\leqq n}|x_i|$). It is not known what the P_λ spaces are and, in particular, the following question is open.

PROBLEM 1. *Is every P_λ space isomorphic to a P_1 space?*

Since any two n-dimensional spaces are isomorphic, the finite dimensional version of Problem 1 should be rephrased. Let X and Y be isomorphic Banach spaces. The Banach–Mazur distance $d(X, Y)$ is defined to be $\inf_T \|T\|\,\|T^{-1}\|$ where the inf is taken over all invertible operators T from X onto Y. Now we may reformulate the finite dimensional version of Problem 1 as follows:

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PROBLEM 2. *Does there exist a function $\psi(\lambda) \geqq 1$ such that, for every k, if E is a k-dimensional P_λ space then $d(E, l_\infty^k) < \psi(\lambda)$?*

As is well known, every finite dimensional space E is isometric to a subspace of l_∞ and since l_∞ is the closure of a set (directed by inclusion) of finite dimensional subspaces E_α where E_α is isometric to $l_\infty^{d(\alpha)}$ ($d(\alpha) =$ dimension E_α), it is clear that in order to answer Problem 2 it is enough to consider subspaces of the spaces l_∞^n onto which there is a projection of norm smaller than λ. Since $(l_\infty^n)^* = l_1^n$ (= the space of all n-tuples $y = (y_1, \cdots, y_n)$ of real numbers with $\|y\| = \sum_{i=1}^n |y_i|$) a duality argument shows that Problem 2 is equivalent to the following.

PROBLEM 3. *Does there exist a function $\psi(\lambda) \geqq 1$ such that for any integers $n > k > 0$ and any projection Q of l_1^n onto a k-dimensional subspace E with $\|Q\| < \lambda$ it is true that $d(E, l_1^k) < \psi(\lambda)$?*

Being unable to solve Problem 3, we restricted our attention to the range of a projection P of l_1^n with norm $\|P\|$ close to 1. In this case we can prove that the range $P(l_1^n)$ is "close" to an l_1 space in the following sense:

THEOREM. *There exist a positive ε_0 and a positive function $\phi(\varepsilon)$ defined for $0 < \varepsilon < \varepsilon_0$ with $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ such that if k and n are any integers satisfying $n > k > 0$ and Q is a projection of l_1^n onto a k dimensional subspace E of l_1^n with $\|Q\| \leqq 1 + \varepsilon$ then there is an integer $h \geqq k(1 - \phi(\varepsilon))$ and an h dimensional subspace F of E with $d(F, l_1^h) \leqq 1 + \phi(\varepsilon)$. Moreover, there is a projection P of l_1^n onto F with $\|P\| \leqq 1 + \phi(\varepsilon)$.*

Recently J. Bourgain [1] has proved that there exist positive functions $s(\lambda)$ and $t(\lambda)$ such that for every n, every n dimensional P_λ space contains a k dimensional subspace F with $d(F, l_\infty^k) \leqq s(\lambda)$ and $k \geqq nt(\lambda)$. A short proof of the same fact was shown to us by W. B. Johnson. However it seems that their methods do not yield the above Theorem.

2. Preliminaries

The special case of the Theorem where $\|Q\| = 1$ is well known and follows from the characterization of P_1 spaces mentioned in the introduction. We will start by giving a simple proof of this special case. The proof uses only trivial convexity arguments. These arguments will be generalized to provide a proof of the Theorem.

LEMMA 1. *Let $n > k > 0$ and let Q be a projection of l_1^n onto a k dimensional subspace E of l_1^n with $\|Q\| = 1$. Then E is isometric to l_1^k.*

Let (a_{ij}) be the matrix representing Q with respect to this basis, i.e., $Qe_i = \Sigma_{j=1}^n a_{ji}e_j$ for all $1 \leqq i \leqq n$. Let $x_1, x_2, \cdots, x_k$ be k linearly independent extremal points of the unit ball of E (such points exist by the Krein–Milman theorem). Clearly, for each $1 \leqq m \leqq k$, x_m is an image of an extremal point of the unit ball of l_1^n. Let $1 \leqq m \leqq k$ and let $x_m = \pm Qe_{i(m)}$, then $\pm x_m = Qe_{i(m)} = \Sigma_{j=1}^n a_{ji(m)}e_j = \Sigma_{j=1}^n a_{ji(m)}Qe_j$ and because x_m is an extremal point and $\Sigma_{j=1}^n |a_{ji(m)}| \leqq \|Q\| = 1$ we have that $\pm x_m = Qe_{i(m)} = \pm Qe_j$ for all j for which $a_{ji(m)} \neq 0$. Hence there is a subset A_m of $\{1, 2, \cdots, n\}$ such that $j \in A_m$ if and only if $a_{ji(m)} \neq 0$ and $Qe_j = \pm Qe_{i(m)} = \pm x_m$. The sets $A_1, A_2, \cdots, A_k$ are clearly pairwise disjoint and $x_m = \pm \Sigma_{j\in A_m} a_{ji(m)}e_j$ for each $1 \leqq m \leqq k$ hence $\operatorname{span}\{x_m\}_{m=1}^k = E$ is isometric to l_1^k. This proves Lemma 1.

The above convexity argument can be generalized to the case where "almost" convex combinations of elements of l_1^n replace convex combinations and certain "maximal" vectors play the role of extremal points of the unit ball of l_1^n. The next two easy lemmas are extensions of convexity arguments in the above mentioned direction.

LEMMA 2. *Let $\frac{1}{2} > \eta > 0$ and $\frac{1}{3}\eta > \theta > 0$. Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be numbers where $x_i \geqq 0$ for all i, $\Sigma_{i=1}^n x_i \leqq 1 + \theta$, $y \leqq \Sigma x_i y_i$ and the following conditions are satisfied: for each i, $y_i \leqq 1 + \theta$ and $1 - \theta \leqq y \leqq 1 + \theta$. Let $A = \{i : y_i \geqq 1 - \eta\}$. Then $\Sigma_{i\in A} x_i > 1 - 3\theta\eta^{-1}$. If $\eta \geqq 3\theta^{1/2}$ then $\Sigma_{i\in A} x_i \geqq 1 - \theta^{1/2}$.*

PROOF. Note that Lemma 1 is a trivial convexity argument if $\theta = \eta = 0$. We have that

$$1 - \theta < y = \sum_{i=1}^n x_i y_i = \sum_{i\in A} x_i y_i + \sum_{i\in A^c} x_i y_i \leqq (1+\theta) \sum_{i\in A} x_i + (1-\eta) \sum_{i\in A^c} x_i$$

$$\leqq (1+\theta)\Big(\sum_{i\in A} x_i\Big) + (1-\eta)\Big[1 + \theta - \Big(\sum_{i\in A} x_i\Big)\Big] = (1-\eta)(1+\theta) + (\theta+\eta)\Big(\sum_{i\in A} x_i\Big).$$

Hence $1 - \theta - (1-\eta)(1+\theta) \leqq (\theta+\eta)(\Sigma_{i\in A} x_i)$ and therefore $\eta - 2\theta \leqq (\eta+\theta)(\Sigma_{i\in A} x_i)$. It follows that $\Sigma_{i\in A} x_i \geqq (1 + \theta/\eta)^{-1}(1 - 2(\theta/\eta)) \geqq (1 - \theta/\eta)(1 - 2\theta/\eta) \geqq 1 - 3\theta/\eta$. If $\eta \geqq 3\theta^{1/2}$ we get that $\Sigma_{i\in A} x_i \geqq 1 - \theta^{1/2}$.

The next lemma is an extension of Lemma 2 to the m dimensional case. It expresses in a quantitative way the fact that certain "almost maximal" vectors v in l_1^m behave like extreme points in the following sense: whenever v is an "almost" convex combination of certain vectors v_i (with $\|v_i\|$ close to 1) then "most" of the v_i's are close to v.

Lemma 3. *Let* $0<\theta<8^{-4}$, *let* $v=\sum_{j=1}^m a_je_j$ *and* $v_i=\sum_{j=1}^m a_{ji}e_j$, $1\leqq i\leqq n$ *be elements of* l_1^m *where* $\{e_j\}_{j=1}^m$ *denotes the unit vector basis. Assume that* $v=\sum_{i=1}^n b_iv_i$ *where* $\sum_{i=1}^n |b_i|\leqq 1+\theta$ *and that* $\|v_i\|\leqq 1+\theta$ *for all* $1\leqq i\leqq n$, *and let* $1-\theta\leqq \|v\|\leqq 1+\theta$. *Let* $M_j=\max_{1\leqq i\leqq n}|a_{ji}|$ *and assume that for each* $1\leqq j\leqq m$, $(1+\theta)|a_j|>M_j$.

Finally, let $B=\{i:\|v-v_i\|<\mu$ *or* $\|v+v_i\|<\mu\}$ *where* $\mu\geqq 8\theta^{1/4}$. *Then* $\sum_{i\in B}|b_i|>1-\theta^{1/2}$.

Proof. We may assume without loss of generality that $a_j\geqq 0$ for all $1\leqq j\leqq m$ (otherwise we replace e_j by $-e_j$). We may also assume that $b_i\geqq 0$ for all i. (Indeed, if $b_i<0$ we may replace it by $-b_i$ and replace v_i by $-v_i$.) Let $4^{-1}>\phi\geqq 2\theta^{1/4}$ and let $C_i=\{j:a_{ji}\leqq a_j(1-\phi)\}$ and $B_0=\{i:\sum_{j\in C_i}a_j<\phi\}$. Then $B_0\subset B$. Indeed, for $i\in B_0$ we have that $\sum_{j\in C_i^c}a_j>1+\theta-\phi$ hence

$$\begin{aligned}\|v-v_i\| &= \sum_{j=1}^m |a_j-a_{ji}| = \sum_{j\in C_i}|a_j-a_{ji}| + \sum_{j\in C_i^c}|a_j-a_{ji}|\\ &\leqq \sum_{j\in C_i}a_j+\sum_{j\in C_i}|a_{ji}|+\phi\sum_{j\in C_i^c}a_j\\ &\leqq \sum_{j\in C_i}a_j+(1+\theta)\sum_{j\in C_i}a_j+\phi(1+\theta)\\ &\leqq \phi(2+\theta)+\phi(1+\theta)=\phi(3+2\theta)<4\phi.\end{aligned}$$

Hence $B_0\subset B$. It remains to estimate $\sum_{i\in B_0}b_i$. First note that if $i\notin B_0$ then $\sum_{j\in C_i^c}a_j\leqq 1+\theta-\phi$ and so

$$\begin{aligned}\sum_{j=1}^m a_{ji} &= \sum_{j\in C_i}a_{ji}+\sum_{j\in C_i^c}a_{ji}\leqq (1-\phi)\sum_{j\in C_i}a_j+(1+\theta)\sum_{j\in C_i^c}a_j\\ &\leqq (1-\phi)\sum_{j=1}^m a_j+(\phi+\theta)\sum_{j\in C_i^c}a_j\\ &\leqq (1-\phi)(1+\theta)+(\phi+\theta)(1+\theta-\phi)\\ &=(1+\theta)^2-\phi\theta-\phi^2\leqq (1+\theta)^2-\phi^2.\end{aligned}$$

Now use Lemma 2 with $x_i=b_i$, $y_i=\sum_{j=1}^m a_{ji}$, $y=\sum_{j=1}^m a_j$ and $\eta=\phi^2-2\theta-\theta^2$. Then we get that $\sum_{i\in B_0}b_i>1-3\theta/\eta$. Since $4^{-1}>\phi>0$ is such that $\phi^2>3\theta^{1/2}+2\theta+\theta^2$ we get that $\eta=\phi^2-2\theta-\theta^2>3\theta^{1/2}$ hence, by Lemma 1, $\sum_{i\in B}b_i\geqq\sum_{i\in B_0}b_i\geqq 1-\theta^{1/2}$. This proves Lemma 3.

REMARK 4. As a consequence of Lemma 3 we get the following more general version of Lemma 3:

If, instead of assuming that $(1+\theta)|a_j| > M_j$ for all $1 \leqq j \leqq m$, we assume that the inequality holds only for $1 \leqq j \leqq \bar{m} < m$ and $\Sigma_{j=1}^{\bar{m}} |a_j| > 1 - \bar{\theta}$ where $\bar{\theta} > 0$ then we may put $\bar{v} = \Sigma_{j=1}^{\bar{m}} a_j e_j$, $\bar{v}_i = \Sigma_{j=1}^{\bar{m}} a_{ji} e_j$, $\bar{\phi} \geqq 2(\bar{\theta})^{1/2}$ and $\bar{B} = \{i : \|\bar{v} - \bar{v}_i\| < \bar{\mu}$ or $\|\bar{v} + \bar{v}_i\| < \bar{\mu}\}$ where $\bar{\mu} = 4\bar{\phi}$. Then, by Lemma 3, if $0 < \bar{\theta} < 8^{-4}$ we get that $\Sigma_{i \in \bar{B}} b_i > 1 - \bar{\theta}^{1/2}$. However, for each $i \in \bar{B}$ we have that

$$\begin{aligned} \|v - v_i\| &= \|\bar{v} - \bar{v}_i\| + \sum_{j=\bar{m}+1}^{m} |a_j - a_{ji}| \leqq \bar{\mu} + \sum_{j=\bar{m}+1}^{m} |a_j| + \sum_{j=\bar{m}+1}^{m} |a_{ji}| \\ &\leqq \bar{\mu} + \bar{\theta} + \theta + (1 + \theta - \|\bar{v}_i\|) \\ &\leqq \bar{\mu} + \bar{\theta} + \theta + 1 + \theta - (1 - \bar{\theta} - \bar{\mu}) \\ &= 2\bar{\mu} + 2\bar{\theta} + 2\theta. \end{aligned}$$

It follows that if $A = \{i : \|v - v_i\| < 17\bar{\theta}^{1/2}$ or $\|v + v_i\| < 17\bar{\theta}^{1/2}\}$ and $\bar{\theta}$ is small enough then $\Sigma_{i \in A} |b_i| > 1 - \bar{\theta}^{1/2}$.

We are interested in the *range* of a projection Q with norm close to 1 and not in the projection itself. It happens to be more convenient to work with projections Q on l_1^n for which $\|Qe_i\|$ is close to 1 for $1 \leqq i \leqq n$. The next lemma tells us that, without loss of generality, we may always assume that Q has this desired property.

LEMMA 5. *Let $\{e_i\}_{i=1}^n$ be the unit vector basis of l_1^n, let $0 < \varepsilon < 16^{-2}$ and let P be a projection of l_1^n onto a k dimensional subspace E of l_1^n with $\|P\| < 1 + \varepsilon$. Then there is a subset $A \subset \{1, 2, \cdots, n\}$ and a projection Q of the space $X = \operatorname{span}\{e_i\}_{i \in A}$ onto its subspace F such that $\|Q\| < 1 + 2\varepsilon^{1/2}$ and $d(E, F) < 1 + 2\varepsilon^{1/2}$. Moreover, for each $i \in A$, $\|Qe_i\| \geqq 1 - 8\varepsilon^{1/2}$.*

PROOF. Let $\mu = \varepsilon^{1/2}(1 + \varepsilon)$, let $A = \{i : \|Pe_i\| > 1 - \mu\}$ and let $Pe_i = \Sigma_{j=1}^n a_{ji} e_j$. Then $\Sigma_{j=1}^n |a_{ji}| < 1 + \varepsilon$ and for any $1 \leqq i \leqq n$

$$\begin{aligned} \sum_{j=1}^{n} |a_{ji}| = \|Pe_i\| = \left\| \sum_{j=1}^{n} a_{ji} Pe_j \right\| &\leqq \left\| \sum_{j \in A} a_{ji} Pe_j \right\| + \left\| \sum_{j \in A^c} a_{ji} Pe_j \right\| \\ &\leqq \left(\sum_{j \in A} |a_{ji}| \right)(1 + \varepsilon) + \left(\sum_{j \in A^c} |a_{ji}| \right)(1 - \mu) \\ &\leqq \left(\sum_{j=1}^{n} |a_{ji}| \right)(1 - \mu) + \left(\sum_{j \in A} |a_{ji}| \right)(\mu + \varepsilon), \end{aligned}$$

hence $\mu \, \Sigma_{j=1}^n |a_{ji}| \leqq (\Sigma_{j \in A} |a_{ji}|)(\mu + \varepsilon)$. It follows that

$$\sum_{j\in A^c}|a_{ji}| \leqq \sum_{j=1}^n |a_{ji}| - \sum_{j\in A}|a_{ji}| \leqq \left(\sum_{j=1}^n |a_{ji}|\right)\varepsilon\mu^{-1} \leqq \mu^{-1}\varepsilon(1+\varepsilon) = \varepsilon^{1/2}.$$

Let T be the operator defined on l_1^n by $T(\Sigma_1^n a_i e_i) = \Sigma_{i\in A} a_i e_i$ and let $X = \operatorname{span}\{e_i\}_{i\in A}$. Then clearly, for each $1 \leqq i \leqq n$, $\|TPe_i - Pe_i\| = \Sigma_{j\in A^c}|a_{ji}| < \varepsilon^{1/2}$ and therefore $\|TP - P\| < \varepsilon^{1/2}$.

Let $\tilde{T}$ be the operator defined by $\tilde{T}x = TPx + (I-P)x$. Then $\|\tilde{T} - I\| \leqq \varepsilon^{1/2}$ and therefore $\tilde{T}$ is invertible. Let $Q = \tilde{T}P\tilde{T}^{-1}|_X$ and let $F = \tilde{T}E$. Then $Q^2 = Q$, $QX = F$, $\|Q\| \leqq \|\tilde{T}\|\|P\|\|\tilde{T}^{-1}\| \leqq (1+\varepsilon)[1-\varepsilon^{1/2}]^{-1} \leqq 1 + 2\varepsilon^{1/2}$ and $d(F,E) \leqq \|\tilde{T}^{-1}\| \leqq (1-\varepsilon^{1/2})^{-1} \leqq 1+2\varepsilon^{1/2}$. Moreover, for each $i \in A$,

$$\|Qe_i\| \geqq \|Pe_i\| - \|P - Q\| \geqq 1 - \mu - (\|P\|\|I - \tilde{T}\| + \|\tilde{T}P\|\|I - \tilde{T}^{-1}\|)$$

$$\geqq 1 - 2\varepsilon^{1/2} - [(1+\varepsilon)\varepsilon^{1/2} + (1+\varepsilon)(1+\varepsilon^{1/2})\|I - \tilde{T}^{-1}\|].$$

But $\|I - \tilde{T}^{-1}\| = \|\tilde{T}^{-1}\tilde{T} - \tilde{T}^{-1}\| \leqq \|\tilde{T}^{-1}\|\|I - \tilde{T}\| \leqq (1-\varepsilon^{1/2})^{-1}\varepsilon^{1/2} \leqq (1+2\varepsilon^{1/2})\varepsilon^{1/2}$ and therefore $\|Qe_i\| \geqq 1 - 8\varepsilon^{1/2}$. This proves Lemma 5.

3. Proof of the Theorem

In view of Lemma 5, in order to prove the theorem it suffices to prove the following.

PROPOSITION. *There exist a positive ε_0 and a positive function $\beta(\varepsilon)$ defined for $0 < \varepsilon < \varepsilon_0$ with $\lim_{\varepsilon\to 0}\beta(\varepsilon) = 0$ such that for every $n > k > 0$, if Q is a projection of l_1^n onto a k dimensional subspace E of l_1^n with $\|Q\| \leqq 1 + \varepsilon$ and if $\|Qe_i\| \geqq 1 - \varepsilon$ for any basis element e_i, $1 \leqq i \leqq n$, then there is an integer $h \geqq k(1-\beta(\varepsilon))$ and a subset $A = \{i(1), i(2), \cdots, i(h)\} \subset \{1, 2, \cdots, n\}$ and there exist pairwise disjoint subsets $A_1, A_2, \cdots, A_h$ of $\{1, 2, \cdots, n\}$ such that for any $1 \leqq m \leqq h$, if $Qe_{i(m)} = \Sigma_{j=1}^n a_{ji(m)}e_j$, then $\Sigma_{j\in A_m}|a_{ji(m)}| > 1 - \beta(\varepsilon)$. Hence E contains an h dimensional subspace $F = \operatorname{span}\{Qe_{i(m)}\}_{m=1}^h$ with $d(F, l_1^h) < (1-\beta(\varepsilon))^{-1}(1+\beta(\varepsilon))$. Moreover, there is a projection P of l_1^n onto F with $\|P\| \leqq (1 - \varepsilon - \beta(\varepsilon))^{-1}(1 + \varepsilon + \beta(\varepsilon))$.*

PROOF OF THE PROPOSITION. Let the projection Q be represented by the matrix $M = (a_{ij})$, i.e., for each basis element e_i, $Qe_i = \Sigma_{j=1}^n a_{ji}e_j$ and let $M_j = \max_{1\leqq i\leqq n}|a_{ji}|$. We will need estimates for $\Sigma_{j=1}^n M_j$. To obtain the upper estimate, let $\|Q\|_\wedge$ denote the nuclear norm of Q, i.e., $\|Q\|_\wedge = \inf \Sigma_j \|f_j\|\|x_j\|$ where the inf is taken over all representations of Q of the form $Qx = \Sigma_j f_j(x)x_j$ (for all $x \in l_1^n$) with $x_j \in l_1^n$ and $f_j \in l_1^{n*} = l_\infty^n$. Since $(e_j)_{j=1}^n$ is the unit vector basis of l_1^n clearly $\|Q\|_\wedge = \Sigma_{j=1}^n \|f_j\|$ where f_j are the functionals for which $Qx = \Sigma_{j=1}^n f_j(x)e_j$ for all $x \in l_1^n$, i.e., $f_j = (a_{j1}, a_{j2}, \cdots, a_{jn}) \in l_\infty^n$. But $\|f_j\| = \max_{1\leqq i\leqq n}|a_{ji}| = M_j$ and thus

$\Sigma_{j=1}^n M_j = \|Q\|_\Lambda$. On the other hand, because E is k-dimensional, Q has a representation $Qx = \Sigma_{j=1}^k Q^* g_j(x) y_j$ where $(y_j, g_j)_{j=1}^k$ is an Auerbach basis of E, that is, $y_j \in E$, $g_j \in E^*$, $g_j(y_i) = \delta_{i,j}$ and $\|y_j\| = \|g_j\| = 1$ for all $1 \leqq j \leqq n$. It follows that

$$\sum_{j=1}^n M_j = \|Q\|_\Lambda \leqq \|Q\| \sum_{j=1}^k \|g_j\| \|y_j\| \leqq k\|Q\| \leqq k(1+\varepsilon).$$

Hence $k = \text{trace}(Q) \leqq \Sigma_{j=1}^n M_j \leqq k(1+\varepsilon)$.

Let $\Omega = \{(j,i): i,j = 1,2,\cdots,n\}$ and define a probability measure P on Ω by $P(j,i) = |a_{ij}| M_j (\Sigma_{p=1}^n a_{pj})^{-1} (\Sigma_{q=1}^n M_q)^{-1}$. It follows from the above estimates that

$$|a_{ij}| M_j (1+\varepsilon)^{-2} k^{-1} \leqq P(j,i) \leqq |a_{ij}| M_j (1-\varepsilon)^{-1} k^{-1}. \tag{1}$$

Let $\delta = \varepsilon^{1/2}$ and for each $1 \leqq j \leqq n$ let $\Gamma_j = \{(j,i): |a_{ji}| > M_j(1-\delta)\}$. The set Γ_j is a subset of the jth row of Ω. We would like to compute $P(\bigcup_{j=1}^n \Gamma_j)$. To do this, first note that for each $1 \leqq j \leqq n$

$$\begin{aligned} a_{jj} = \sum_{i=1}^n a_{ij} a_{ji} &\leqq \sum_{(j,i)\in\Gamma_j} a_{ij} a_{ji} + \sum_{(j,i)\in\Gamma_j^c} a_{ij} a_{ji} \\ &\leqq \Big(\sum_{(j,i)\in\Gamma_j} |a_{ij}| \Big) M_j + \Big(\sum_{(j,i)\in\Gamma_j^c} |a_{ij}| \Big) M_j (1-\delta) \\ &\leqq \Big(\sum_{(j,i)\in\Gamma_j} |a_{ij}| \Big) M_j + \Big[(1+\varepsilon) - \Big(\sum_{(j,i)\in\Gamma_j} |a_{ij}| \Big) \Big] M_j (1-\delta) \\ &\leqq M_j (1+\varepsilon)(1-\delta) + \Big(\sum_{(j,i)\in\Gamma_j} |a_{ij}| \Big) M_j \delta. \end{aligned}$$

Taking the sum over j we get that

$$k = \sum_{j=1}^n a_{jj} \leqq (1+\varepsilon)(1-\delta) \sum_{j=1}^n M_j + \delta \sum_{j=1}^n M_j \Big(\sum_{(j,i)\in\Gamma_j} |a_{ij}| \Big).$$

Since $\Sigma_{j=1}^n M_j \leqq k(1+\varepsilon)$ it follows that

$$k[1 - (1+\varepsilon)^2 (1-\delta)] \leqq \delta \sum_{j=1}^n M_j \Big(\sum_{(j,i)\in\Gamma_j} |a_{ij}| \Big)$$

and therefore, by (1),

$$1 - 3\delta^{-1}\varepsilon \leqq k^{-1} \sum_{j=1}^n M_j \Big(\sum_{(j,i)\in\Gamma_j} |a_{ij}| \Big) \leqq (1+\varepsilon)^2 P\Big(\bigcup_{j=1}^n \Gamma_j \Big).$$

It follows that

$$P\left(\bigcup_{j=1}^{n} \Gamma_j\right) \geqq (1-3\varepsilon^{1/2})(1+\varepsilon)^{-2} \geqq 1-4\varepsilon^{1/2} \tag{2}$$

if ε is small enough.

Now let $\Gamma_j^* = \{(i,j):(j,i)\in\Gamma_j\}$, i.e., Γ_j^* is a subset of the jth column of Ω. We need an estimate on $P(\bigcup_{j=1}^n \Gamma_j^*)$. For each $(i,j)\in\Gamma_j^*$ we have that $(j,i)\in\Gamma_i$ and hence $|a_{ji}|>M_j(1-\delta)$. It follows from (1) that

$$\begin{aligned} P(i,j) &\geqq (1+\varepsilon)^{-2}k^{-1}|a_{ji}|M_i \geqq (1+\varepsilon)^{-2}k^{-1}(1-\delta)M_jM_i \\ &\geqq (1+\varepsilon)^{-2}k^{-1}(1-\delta)M_j|a_{ij}| \geqq (1+\varepsilon)^{-2}(1-\delta)(1-\varepsilon)P(j,i) \geqq (1-4\varepsilon^{1/2})P(j,i) \end{aligned} \tag{3}$$

if ε is small enough. Combining (2) and (3) we get that

$$P\left(\bigcup_{j=1}^{n} \Gamma_j^*\right) \geqq (1-4\varepsilon^{1/2})P\left(\bigcup_{j=1}^{n} \Gamma_j\right) \geqq (1-4\varepsilon^{1/2})^2 \geqq 1-9\varepsilon^{1/2} \tag{4}$$

Let $\Delta = (\bigcup_{j=1}^n \Gamma_j)\cap(\bigcup_{j=1}^n \Gamma_j^*)$, then $(i,j)\in\Delta$ if and only if $(j,i)\in\Delta$ and

$$1-P(\Delta) \leqq 1-P\left(\bigcup_{j=1}^{n} \Gamma_j\right)+1-P\left(\bigcup_{j=1}^{n} \Gamma_j^*\right) = 15\varepsilon^{1/2} \tag{5}$$

if ε is small enough.

Let $\Delta_j = \{i:(i,j)\in\Delta\} = \{i:(j,i)\in\Delta\}$. We know that if $i\in\Delta_j$ then $|a_{ij}|> M_i(1-\delta)$. We are interested in those columns of the matrix (a_{ij}) for which $\Sigma_{i\in\Delta_j}|a_{ij}|$ is close to 1, and we would like to estimate the number of these columns. It follows from (5) that

$$1-15\varepsilon^{1/2} \leqq P(\Delta) = \sum_{j=1}^{n}\sum_{i\in\Delta_j} P(j,i) \leqq (1-\varepsilon)^{-1}k^{-1}\sum_{j=1}^{n}\sum_{i\in\Delta_j}|a_{ij}|M_j$$

hence, for small $\varepsilon>0$, we have that

$$1-20\varepsilon^{1/2} \leqq \sum_{j=1}^{n}(k^{-1}M_j)\left(\sum_{i\in\Delta_j}|a_{ij}|\right). \tag{6}$$

Now by Lemma 2 with $\theta = 20\varepsilon^{1/2}$, $x_j = k^{-1}M_j$, $y_j = \Sigma_{i\in\Delta_j}|a_{ij}|$ and $\eta = 3\theta^{1/2}$ we get a subset $A\subset\{1,2,\cdots,n\}$ such that for each $j\in A$, $\Sigma_{i\in\Delta_j}|a_{ij}|>1-\eta$ and $\Sigma_{j\in A}k^{-1}M_j \geqq 1-\theta^{1/2}$. Recall that for $i\in\Delta_j$, $|a_{ij}|\geqq M_i(1-\delta)$ and hence $Qe_q = \Sigma_{j=1}^n a_{jq}e_j$ with $q\in A$ has the property that "most" of its components satisfy $|a_{jq}|\geqq M_j(1-\delta)$. Consequently, since $Qe_q = \Sigma_{i=1}^n a_{iq}e_i = Q^2e_q = \Sigma_{i=1}^n a_{iq}Qe_i$ we get, in view of Remark 4, that there is a subset A_q of $\{1,2,\cdots,n\}$ such that $\Sigma_{i\in A_q}|a_{i,q}|>1-\bar\theta^{1/2}$ and for each $i\in A_q$ either $\|Qe_q-Qe_i\|<17\bar\theta^{1/2}$ or $\|Qe_q+Qe_i\|<17\bar\theta^{1/2}$ (we use Remark 4 with $v_i = Qe_i$, $\theta=(1-\delta)^{-1}-1$ and

$\bar\theta = \eta = 3\theta^{1/2}$). Let $\alpha = 17\bar\theta^{1/2}$. If $p \in A$ then there is a subset A_p of $\{1,2,\cdots,n\}$ such that $\Sigma_{i\in A_p}|a_{ip}| > 1-\bar\theta^{1/2}$ and for each $i \in A_p$ either $\|Qe_p - Qe_i\| < \alpha$ or $\|Qe_p + Qe_i\| < \alpha$. Hence, if for both signs $\pm$, $\|Qe_p \pm Qe_q\| > 3\alpha$ then A_p and A_q are disjoint sets.

Thus for each $p \in A$ either for one sign $\pm Qe_p$ is within distance 3α from Qe_q or for both signs $\|Qe_p \pm Qe_q\| \geqq 2-2\bar\theta^{1/2}$. It follows that there is a natural number $h \geqq 1$, there are h integers $1 \leqq i(1) < i(2) < \cdots < i(h) \leqq n$ and there are h pairwise disjoint subsets $\{A_m\}_{m=1}^h$ such that $\Sigma_{i\in A_m}|a_{i\,i(m)}| > 1-\bar\theta^{1/2}$ for all $1 \leqq m \leqq h$ (hence $\{Qe_{i(m)}\}_{m=1}^h$ are vectors with "almost" disjoint supports). Moreover, if $j \in A$ then there is one integer $p(j)$, $1 \leqq p(j) \leqq h$ such that $\|Qe_j \pm Qe_{i_{p(j)}}\| \leqq 3\alpha$ for one choice of the sign. We want h to be as close to k ($=\dim E$) as possible. Unfortunately, so far we have no reason to believe even that $h > 1$. The estimate on h will be achieved by arguments concerning the trace. Let R be an operator on l_1^n defined by the matrix $(b_{i,j})$, i.e., $Re_j = \Sigma_{i=1}^n b_{i,j}e_i$, where $b_{ij} = a_{ij}$ if $j \notin A$ while, for $j \in A$, $b_{ij} = \pm a_{i\,p(j)}$, where the integer $p(j)$ and the sign are chosen so that $\|Qe_j \pm Qe_{i_{p(j)}}\| \leqq 3\alpha$. It follows that $\|Q-R\| \leqq 3\alpha$.

Now note that $Re_i \in E$ for all $1 \leqq i \leqq n$, hence the range of R is contained in the k dimensional subspace E and therefore $|\mathrm{trace}(Q-R)| \leqq \|Q-R\|k \leqq 3\alpha k$. It follows that $\mathrm{trace}\,R \geqq \mathrm{trace}\,Q - |\mathrm{trace}(Q-R)| \geqq k(1-3\alpha)$. On the other hand,

$$\mathrm{trace}\,R = \sum_{j=1}^n b_{jj} = \sum_{j\notin A} a_{jj} + \sum_{m=1}^h \sum_{p(j)=i(m)} \pm a_{ji(m)} \leqq \sum_{j\notin A} M_j + \sum_{m=1}^h \|Qe_{i(m)}\|$$

$$\leqq \sum_{j=1}^n M_j - \sum_{j\in A} M_j + h(1+\varepsilon) \leqq k(1+\varepsilon) - k(1-\theta^{1/2}) + h(1+\varepsilon).$$

It follows that $h \geqq k[1-3\alpha-(1+\varepsilon)+(1-\theta^{1/2})](1+\varepsilon)^{-1}$, clearly $[1-3\alpha-(1+\varepsilon)+(1-\theta^{1/2})](1+\varepsilon)^{-1} \to 1$ as $\varepsilon \to 0$. The existence of the desired projection follows by a standard perturbation argument. This completes the proof of the Proposition.

REMARK 6. We suspect that one cannot prove the Theorem with $h = k$ by using only the "almost maximal" vectors appearing in the proof above (i.e., vectors $v_i = \Sigma_{j=1}^n a_{ji}e_j$ where, for "most" j, $|a_{ji}| > M_j(1-\delta)$). Indeed, let Q be the projection defined on l_1^{2n} by the matrix $(a_{ij})_{i=1}^{2n}{}_{j=1}^{2n}$ where $a_{ii} = 1$ for $1 \leqq i \leqq n$, $a_{ij} = 0$ for all $1 \leqq i \leqq n$ and $j \neq i$, $a_{ij} = n^{-1}$ for all $n+1 \leqq i, j \leqq 2n$, $a_{ij} = \varepsilon(1-n^{-1})$ if $n+1 \leqq i \leqq 2n$ and $j = i-n$ and $a_{ij} = -\varepsilon n^{-1}$ if $n+1 \leqq i \leqq 2n$ and $j \neq i-n$, $1 \leqq j \leqq n$. In this example E is of dimension $n+1$ and $Qe_i = n^{-1}\Sigma_{j=n+1}^{2n} e_j$ for all $n+1 \leqq i \leqq 2n$ while $M_j = \varepsilon(1-n^{-1})$ for $n+1 \leqq j \leqq 2n$, i.e., the components of

Qe_i are much smaller than the respective maxima M_i in this case ($n+1 \leqq i \leqq 2n$).

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